Cockcroft Institute Lectures

# Electromagnetism

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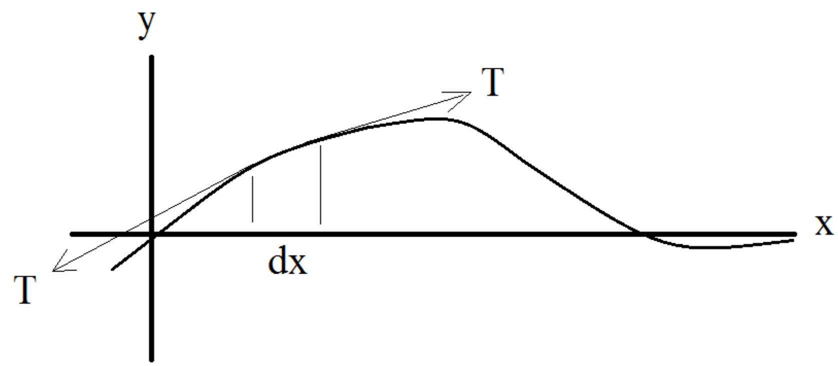
Autumn Term 2013

# Objectives

We shall

- review the maths needed,
- derive Maxwell's equations,
- derive the wave equation,
- derive waveguide modes,
- derive synchrotron radiation,
- review vector potential, and
- derive the Hamiltonian.

In a stretched string, tension is $T$ and mass per unit length is $\rho$. Consider a small section $dx$ that is displaced.




resultant vertical force = mass × vertical acceleration

$$Ty'(x+dx) - Ty'(x) = \rho dx \times \frac{\partial^2 y}{\partial t^2}$$

Divide both sides by $dx$.

$$T\frac{\partial^2 y}{\partial x^2} = \rho\frac{\partial^2 y}{\partial t^2}$$

Solution has a very general form (can check by substution):

$$y = f(x - vt) \text{ where } v = \sqrt{\frac{T}{\rho}}$$

is a solution, where $v = \sqrt{T/\rho}$. This means a function of any shape can travel along the string - just what we expect when we shake one end of a string. $v$ is then the velocity.

For particular frequency $\omega = 2\pi f$ and small displacements, we often assume

$$y = \sin(k(x - vt)) = \sin(kx - \omega t) \text{ where } v = \frac{\omega}{k}$$

In terms of $v$, the equation is

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

In 3D, the equation generalises to

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

where $f(x, y, z)$ is some 3D function which could be pressure, electric field, etc.

$A(x, y, z, t)$ is a function of many variables. The change in $A$ as a result of small changes in the variables is the differential

$$dA = \frac{\partial A}{\partial x} dx + \frac{\partial A}{\partial y} dy + \frac{\partial A}{\partial z} dz + \frac{\partial A}{\partial t} dt$$

If we need the time rate of change of $A$ when all variables are changing, we cannot just take $\partial A / \partial t$. We must divide all of $dA$ by $dt$:

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial z}\frac{dz}{dt} + \frac{\partial A}{\partial t}\frac{dt}{dt}$$

We often use $\dot{x}$ for $dx/dt$, so

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\dot{x} + \frac{\partial A}{\partial y}\dot{y} + \frac{\partial A}{\partial z}\dot{z} + \frac{\partial A}{\partial t}$$

Note that $\frac{dA}{dt}$ and $\frac{\partial A}{\partial t}$ are completely different. In $\frac{\partial A}{\partial t}$, $x$, $y$ and $z$ must be fixed.

Scalar function (or scalar field)

$$\phi(x, y, z) \text{ or } \phi(\mathbf{x})$$

Vector function (or vector field)

$$\mathbf{A}(x, y, z) = \mathbf{i}A_x(x, y, z) + \mathbf{j}A_y(x, y, z) + \mathbf{k}A_z(x, y, z)$$

Gradient

$$\nabla\phi = \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}$$

Curl

$$\nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

Divergence

$$\nabla.\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

Divergence of a curl is zero

$$\nabla.(\nabla \times \mathbf{A}) = 0$$

Curl of a gradient is zero

$$\nabla \times (\nabla \phi) = 0$$

Curl of a curl

$$\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla.\mathbf{E}) - \nabla^2 \mathbf{E}$$

Divergence of cross product

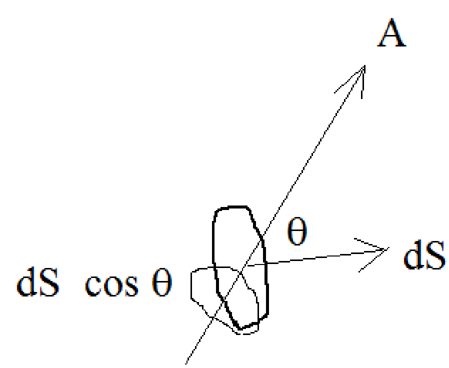
$$\nabla.(\mathbf{E} \times \mathbf{H}) = (\nabla \times \mathbf{E}).\mathbf{H} - (\nabla \times \mathbf{H}).\mathbf{E}$$

To find more, google "vector calculus identities".

You can prove these using the definitions of div, grad and curl.

Relates the vector field in enclosed in a volume to the flux of the field at the surface. Useful for electric and magnetic fields.

Start with physical meaning of divergence.

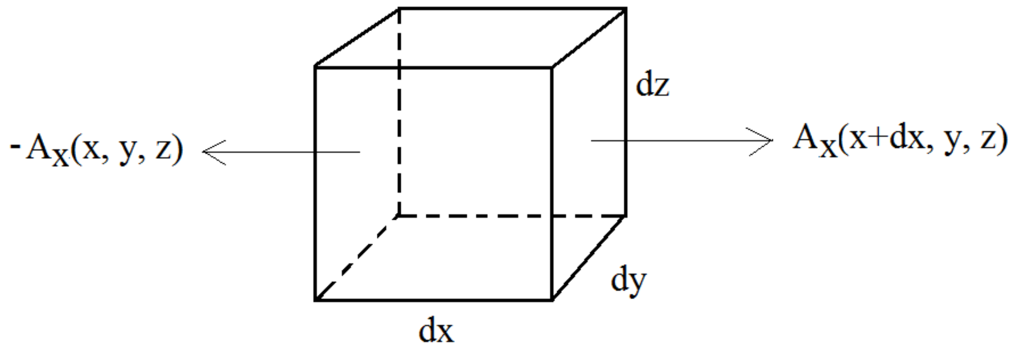



Define flux of a field $\mathbf{A}$ through a small area $d\mathbf{S}$ as

$$\text{flux } = \mathbf{A}.d\mathbf{S}$$

The dot product means the same as projecting $d\mathbf{S}$ in the direction of $\mathbf{A}$ so that

$$\text{flux } = AdS\cos\theta$$




Consider a small volume element $dxdydz$. The sum of flux at two opposite sides along $x$ direction are

$$A_x(x+dx,y,z)dydz - A_x(x,y,z)dydz = \frac{\partial A_x}{dx}dxdydz$$

The total flux over all sides is therefore

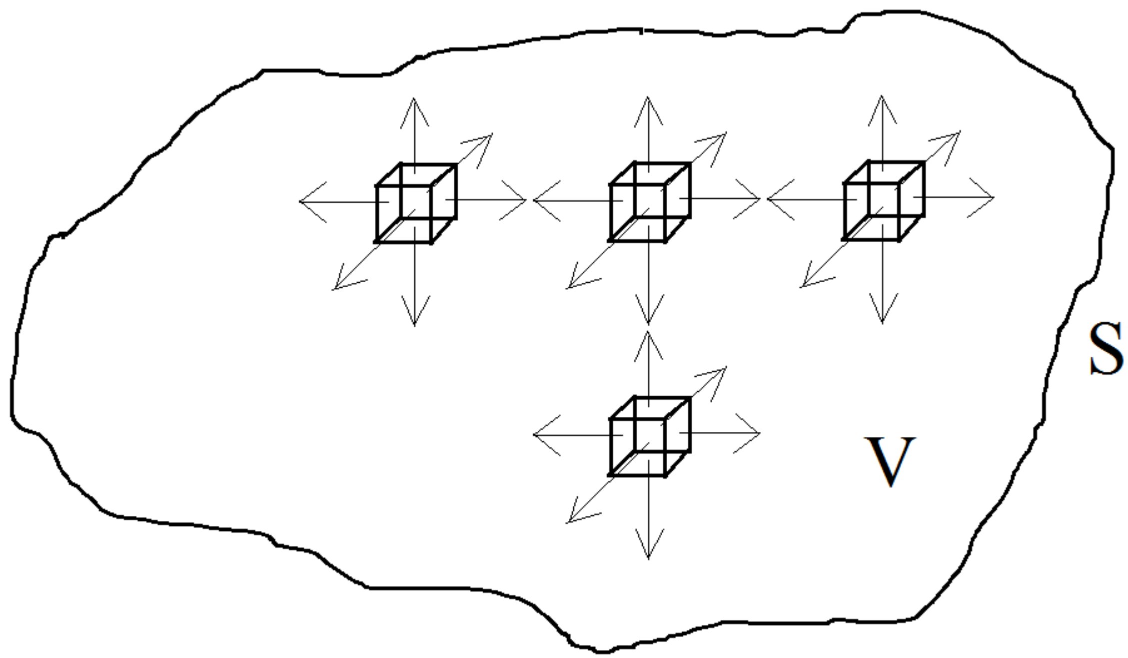
$$\int_{\delta S}\mathbf{A}.d\mathbf{S} = \frac{\partial A_x}{dx}dxdydz + \frac{\partial A_y}{dy}dxdydz + \frac{\partial A_z}{dz}dxdydz \tag{1}$$

This gives the physical meaning of divergence:

$$\frac{\int_{\delta S}\mathbf{A}.d\mathbf{S}}{\delta V} = \nabla.\mathbf{A} \tag{2}$$

Divergence at a point is the net flux coming out of the point per unit volume.

Now imagine a large volume $V$ enclosed by surface $S$. This is in a vector field $\mathbf{A}$. Divide this volume into small volume elements packed together.




Consider the flux of $\mathbf{A}$ from each element. Clearly, fluxes from touching surfaces cancel.

This means that all fluxes cancel except those at the outermost surface $S$.

Recall the flux at each element from Eq. (1)

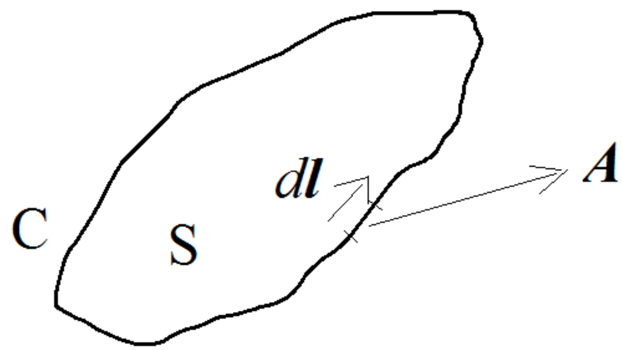
$$\int_{\delta S} \mathbf{A}.d\mathbf{S} = \frac{\partial A_x}{dx} dxdydz + \frac{\partial A_y}{dy} dxdydz + \frac{\partial A_z}{dz} dxdydz$$

Summing over all elements gives

$$\int_S \mathbf{A}.d\mathbf{S} = \int_V \nabla.\mathbf{A} dV$$

This is the divergence theorem.

Consider a loop $C$ in a vector field $\mathbf{A}$. Consider a short segment $d\mathbf{l}$ along $C$.

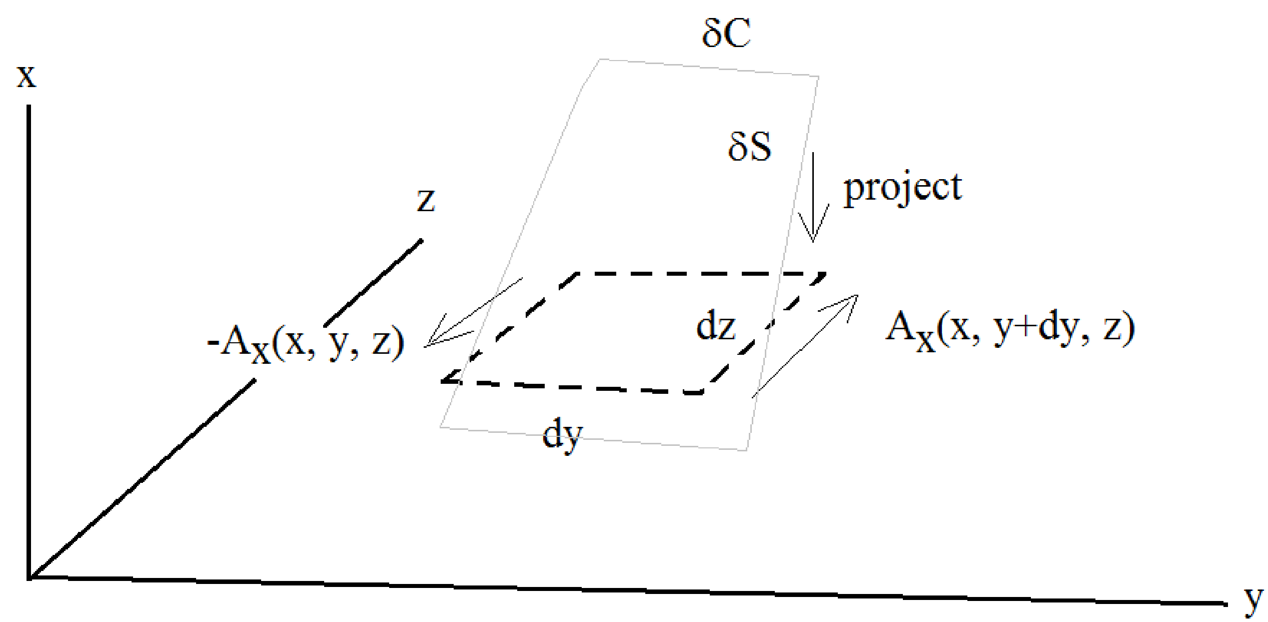



At each segment, take the dot product $\mathbf{A}.d\mathbf{l}$. If we think of $\mathbf{A}$ as force, then $\mathbf{A}.d\mathbf{l}$ is work done. Sum this up over all segments of the loop. This is called the

$$\text{circulation } = \int_C \mathbf{A}.d\mathbf{l}$$

If the loop becomes very small, the circulation approaches the curl component normal to the loop.

To see this, project a small, square loop onto the $yz$ plane through the loop.




Contribution from sides $z$ direction is

$$A_z(x, y + dy, z)dz - A_z(x, y, z)dz = \frac{\partial A_z}{\partial y} dydz$$

Adding the contribution from sides along $y$ direction gives

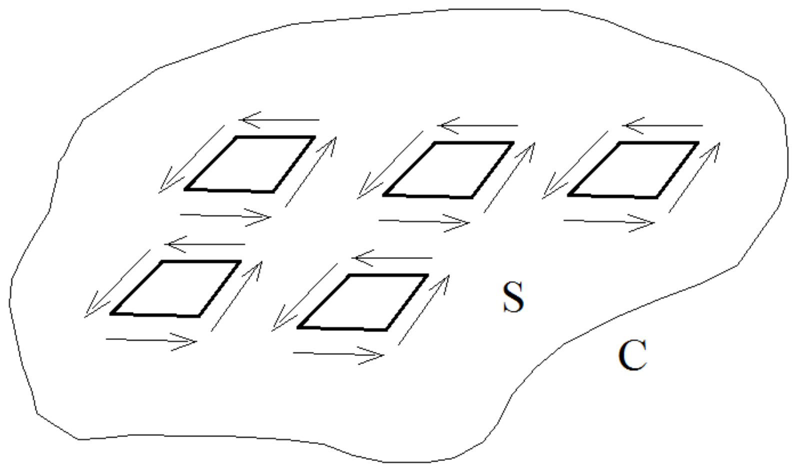
$$\frac{\partial A_z}{\partial y}dydz - \frac{\partial A_y}{\partial z}dzdy = (\nabla \times \mathbf{A})_x dydz$$

which is the $x$ component of curl $\mathbf{A}$.

Adding all 3 components together gives

$$\int_{\delta C} \mathbf{A}.d\mathbf{l} = (\nabla \times \mathbf{A}).\delta\mathbf{S} \tag{3}$$

Now consider a larger loop $C$. Imagine a surface enclosed by this loop. It can be any of the possible surfaces that has $C$ as boundary.



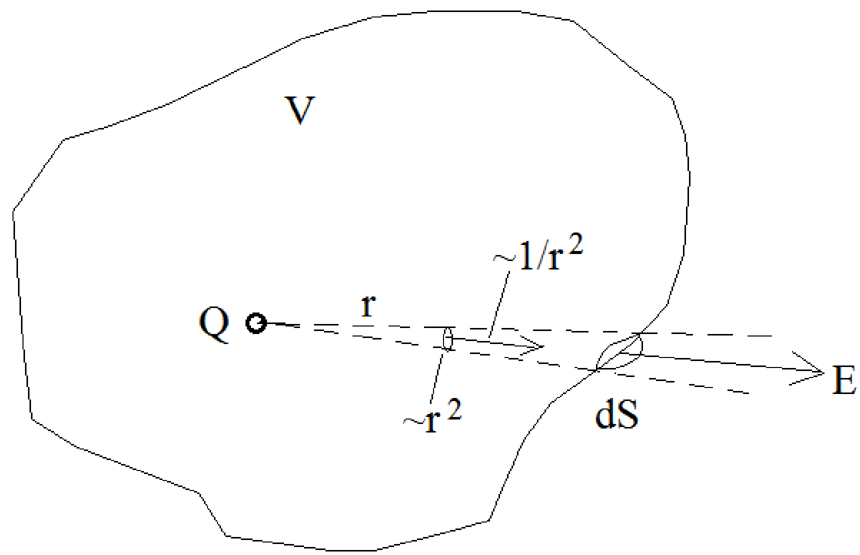

Divide $S$ into small square elements $\delta\mathbf{S}$.

Add up the circulations due to all elements. Contributions from adjacent elements to circulation along a common edge cancel. Only contributions along the outermost $C$ remain.

Summing Eq. (3) over all elements gives Stoke's theorem:

$$\int_C \mathbf{A}.d\mathbf{l} = \int_S (\nabla \times \mathbf{A}).d\mathbf{S}$$

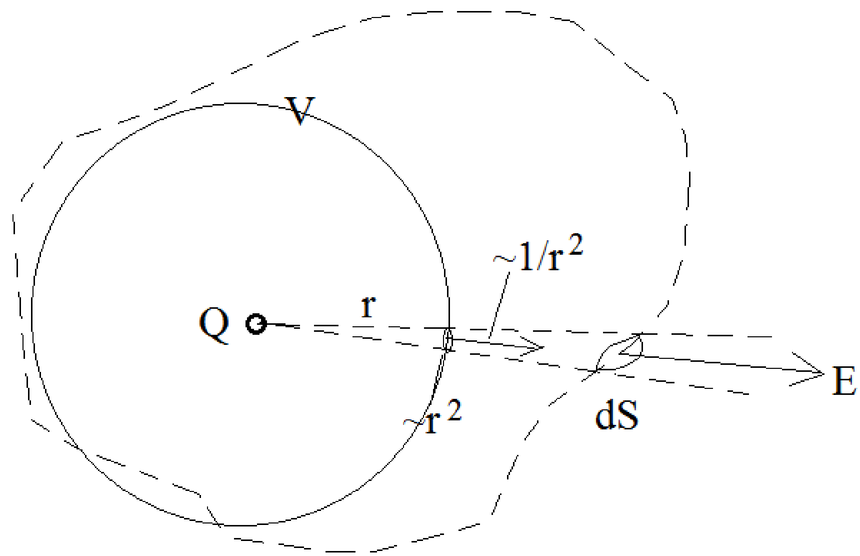
Consider a volume $V$ enclosed in surface $S$. We want to show that the total flux of electric field through $S$ is related to the total charge in $V$.




Suppose there is a point charge $Q$ in $V$. Consider a narrow cone of field from $Q$ crossing the surface at $dS$.

Area $dS$ varies as $r^2$ from $Q$. Field $E$ varies as $1/r^2$ So flux is the same for any surface element in the cone.

Therefore the total flux is equal to that through a sphere centre at $Q$.




So total flux through $S$ is

$$\int_S \mathbf{E}.d\mathbf{S} = \frac{Q}{4\pi\epsilon_0 r^2} \times 4\pi r^2 = \frac{Q}{\epsilon_0}$$

For arbitrary distribution of charges $\rho(\mathbf{r})$, $Q$ generalises to the total charge enclosed:

$$\int_S \mathbf{E}.d\mathbf{S} = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{r})dV$$

Using divergence theorem:

$$\int_V \nabla.\mathbf{E}dV = \frac{1}{\epsilon_0}\int_V \rho(\mathbf{r})dV$$

Because this is true for any volume $V$, the integrands must be equal:

$$\nabla.\mathbf{E} = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

This and the integral above are the two forms of Gauss's law.

A magnet always has north and south pole. The field distribution from each pole is like that of an electric charge. So Gauss's law also applies to magnetic field.

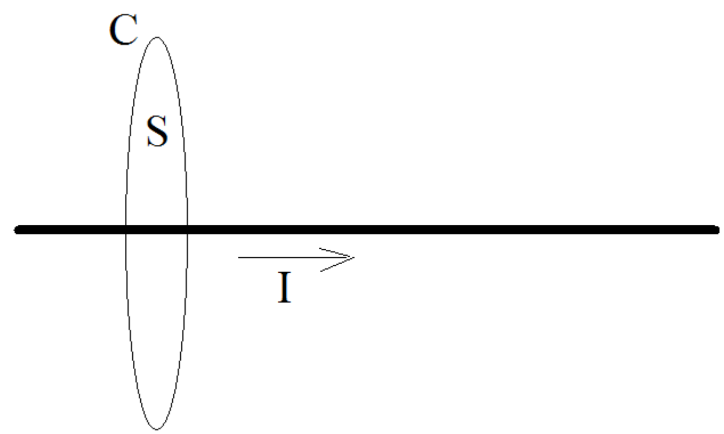
Because north and south poles are like opposite charges, net "magnetic" charge is always zero. Gauss's law for magnetism can be written down from Gauss's law for electricity. Just replace $\mathbf{E}$ by $\mathbf{B}$ and charges by zero:

$$\int_S \mathbf{B}.d\mathbf{S} = 0$$

$$\nabla.\mathbf{B} = 0$$

A current density $I$ through a straight conductor produces a circular magnetic field $B$ around it related by Ampere's law:

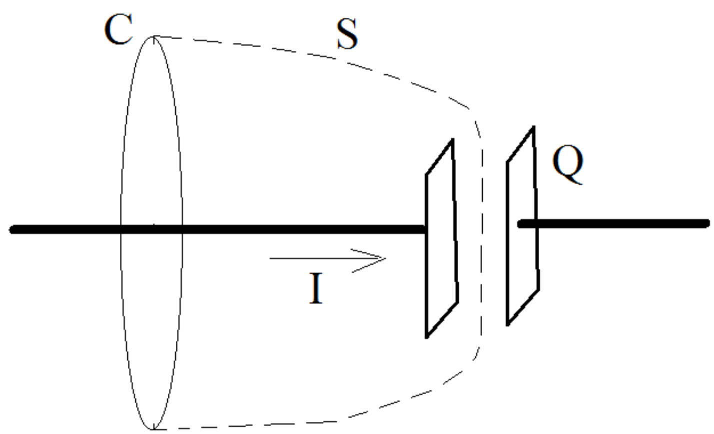
$$\int_C \mathbf{B}.d\mathbf{l} = \mu_0 I$$




Generalising to current density $J$,

$$\int_C \mathbf{B}.d\mathbf{l} = \mu_0 \int_S \mathbf{J}.d\mathbf{S}$$

This is true for a surface through a current. But what if $S$ goes through the empty space of a capacity with no $J$?




Maxwell generalised $J$ to include changing $E$ field between capacitor. $E$ is related to charge $Q$ on capacitor of area $A$ by

$$E = \frac{Q}{\epsilon_0 A} \text{ (From Gauss's law. How?)}$$

This is related to current by

$$I = \frac{dQ}{dt} = \epsilon_0 A \frac{dE}{dt}$$

and to current density by

$$J = \frac{I}{A} = \epsilon_0 \frac{dE}{dt}$$

This is called displacement current. Ampere's law is then generalised to

$$\int_C \mathbf{B}.d\mathbf{l} = \mu_0 \int_S \mathbf{J}.d\mathbf{S} + \epsilon_0\mu_0 \int_S \frac{\partial \mathbf{E}}{\partial t}.d\mathbf{S}$$

Applying Stoke's theorem,

$$\int_S \nabla \times \mathbf{B}.d\mathbf{S} = \mu_0 \int_S \mathbf{J}.d\mathbf{S} + \epsilon_0\mu_0 \int_S \frac{\partial \mathbf{E}}{\partial t}.d\mathbf{S}$$

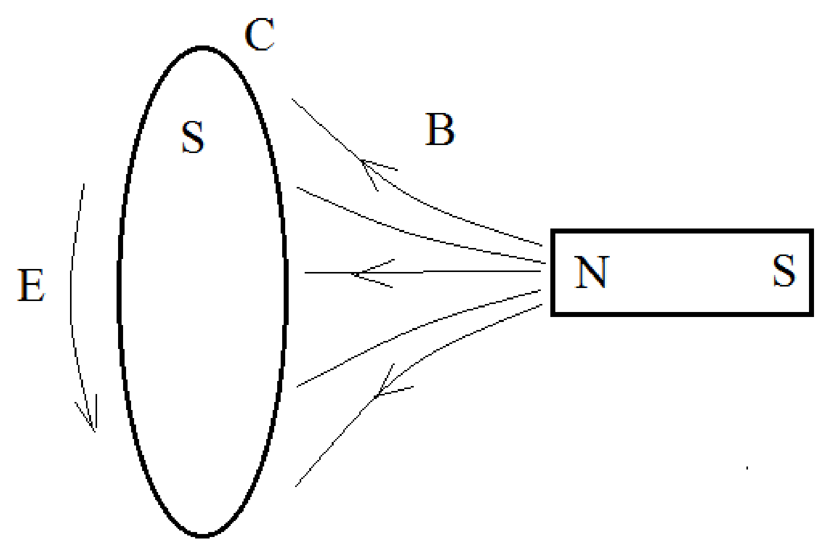
Since this is true for and surface $S$, the integrands on two sides must be equal:

$$\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \epsilon_0\mu_0\frac{\partial \mathbf{E}}{\partial t}$$

This and the above integral equation are the 2 forms of Ampere's law with displacement current.

When the magnetic flux $\Phi$ linking a coil changes, a voltage $V$ is induced round the coil $C$:

$$V = -\frac{d\Phi}{dt}$$




In terms of magnetic field $B$ and electric field $E$:

$$\int_C \mathbf{E}.d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S}$$

Using Stoke's theorem,

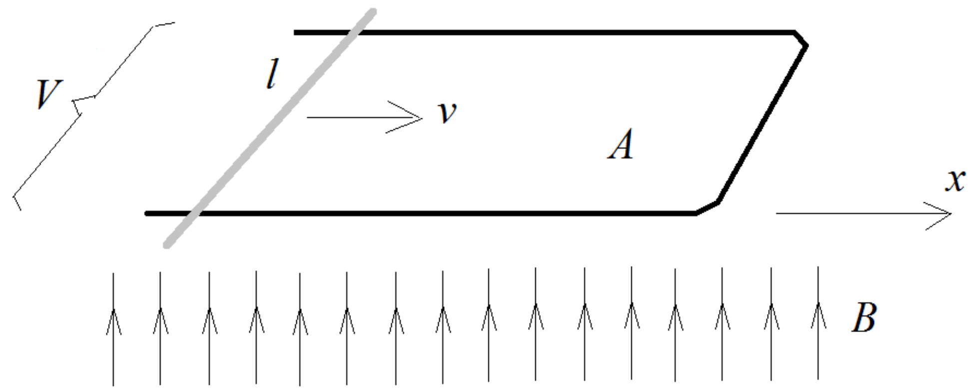
$$\int_S \nabla \times \mathbf{E}.d\mathbf{S} = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S}$$

Since this is true for any surface $S$, the integrands must be equal:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

This is the differential form of Faraday's law.

Consider a horizontal rectangular wire loop with one sliding side and vertical magnetic field through it.

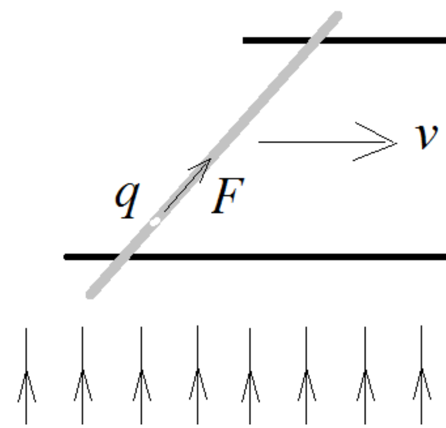



In terms of magnetic field $B$ and electric field As the side moves with velocity $v$, flux changes and voltage $V$ is induced. Faraday's law leads to

$$V = -\frac{d(BA)}{dt} = -\frac{d(Blx)}{dt} = Blv$$

Current induced can be understood as due to forces on electrons in moving side.

$V$ is work done $w$ per unit charge $q$. Work done is due to force $F$ acting on charge over side $l$:




$$w/q = Fl/q = Blv \implies F = qBv$$

If $B$ is at some angle $\theta$ to $v$, only component of $B$ perpendicular to $v$ has effect. So

$$F = qBv\,\mathsf{sin}\,\theta \implies \mathbf{F} = q\mathbf{v} \times \mathbf{B}$$

Including the force from any electric field $E$,

$$\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$$

This is resultant is called the Lorentz force.

Maxwell's equations:

| | |
|---|---|
| Gauss's law | $\nabla.\mathbf{E} = \frac{\rho}{\epsilon_0}$ |
| Gauss's law for magnetism | $\nabla.\mathbf{B} = 0$ |
| Ampere's law | $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$ |
| Faraday's law | $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ |

In a vacuum with no charge and current, these become

| | |
|---|---|
| Gauss's law | $\nabla.\mathbf{E} = 0$ |
| Gauss's law for magnetism | $\nabla.\mathbf{B} = 0$ |
| Ampere's law | $\nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}$ |
| Faraday's law | $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ |

Take the curl of Faraday's law:

$$\nabla \times (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}\nabla \times \mathbf{B}$$

Apply a vector identity to the left and Ampere's law to the right:

$$\nabla(\nabla.\mathbf{E}) - \nabla^2\mathbf{E} = -\epsilon_0\mu_0\frac{\partial}{\partial t}\frac{\partial \mathbf{E}}{\partial t}$$

Using Gauss's law, first term on left is zero. So

$$\nabla^2\mathbf{E} = \epsilon_0\mu_0\frac{\partial^2\mathbf{E}}{\partial t^2}$$

E.g. $x$ component is:

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \epsilon_0\mu_0\frac{\partial^2 E_x}{\partial t^2}$$

Recall the 3D wave equation

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

Clearly, we had a wave equation for electric field. The solution could be any electric field variation that travels with speed $v$. Comparing the equations,

$$v = \frac{1}{\sqrt{\epsilon_0 \mu_0}} = 2.998 \times 10^8 \text{m/s}$$

the speed of light!

1864: This result from Maxwell is strong evidence that light is an electromagnetic wave.

1888: Hertz generated a new (at that time) type of wave which we now call microwave - by using spark gap to create disturbance in electric field.

Any waveform travelling at speed of light is a solution. For wave of a single frequency, consider

$$E_x = E_0 \cos(k_x x - \omega t)$$

For 3D, we can check by substitution that this is also a solution

$$E_x = E_0 \cos(k_x x + k_y y + k_z z - \omega t) = E_0 \cos(\mathbf{k}.\mathbf{r} - \omega t)$$

This is a wave travelling in the $\mathbf{k} = (k_x, k_y, k_z)$ direction. It is real part of the complex form:

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}.\mathbf{r} - \omega t)}$$

where the $E_y$ and $E_z$ are included to form $\mathbf{E}$. Substituting into Faraday's law and doing the differention gives

$$i\mathbf{k} \times \mathbf{E}_0 = i\omega \mathbf{B_0}$$

1. So an electric wave must generate a magnetic wave.
2. The magnetic oscillation must be perpendicular to the electric one.
3. Both magnetic and electric oscillations must be perpendicular to the wave direction $\mathbf{k}$.

This describes the transverse electromagnetic wave in free space.

Waveguides are used to transmit electromagnetic waves with minimal loss. A familiar form is the long, uniform, hollow metallic pipe used for microwave.

Suppose the axis is along $z$. For a fixed frequency, field $\mathbf{H}$ contains factor $e^{-i\omega t}$. Time derivative gains factor $-i\omega$. So Faraday's law becomes:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} = i\omega\mu\mathbf{H} \implies \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_x & E_y & E_z \end{vmatrix} = i\omega\mu\mathbf{H}$$

here $\mu\mathbf{H} = \mathbf{B}$. In full:

$$\begin{aligned} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} &= i\omega\mu H_x \\ -\left(\frac{\partial E_z}{\partial x} - \frac{\partial E_x}{\partial z}\right) &= i\omega\mu H_y \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= i\omega\mu H_z \end{aligned}$$

We drop the subscripts to $\epsilon_0$ and $\mu_0$ so the equations also apply to wave in a medium.

Assume (as trial solution) that spatial variation of $z$ is $e^{ik_z z}$. So

$$
\begin{aligned}
\frac{\partial E_z}{\partial y} - ik_z E_y &= i\omega\mu H_x \\
-\left(\frac{\partial E_z}{\partial x} - ik_z E_x\right) &= i\omega\mu H_y \\
\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} &= i\omega\mu H_z
\end{aligned}
$$

A similar expansion of Ampere's law gives

$$
\begin{aligned}
\frac{\partial H_z}{\partial y} - ik_z H_y &= -i\omega\epsilon E_x \\
-\left(\frac{\partial H_z}{\partial x} - ik_z H_x\right) &= -i\omega\epsilon E_y \\
\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} &= -i\omega\epsilon E_z
\end{aligned}
$$

Solving for transverse components:

$$H_x = -\frac{i}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial y} - k_z\frac{\partial H_z}{\partial x}\right)$$
$$H_y = \frac{i}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial x} + k_z\frac{\partial H_z}{\partial y}\right)$$
$$E_x = \frac{i}{k_c^2}\left(k_z\frac{\partial E_z}{\partial x} + \omega\mu\frac{\partial H_z}{\partial y}\right)$$
$$E_y = -\frac{i}{k_c^2}\left(-k_z\frac{\partial E_z}{\partial y} + \omega\mu\frac{\partial H_z}{\partial x}\right)$$

where $k_c^2 = \omega^2\mu\epsilon - k_z^2$.

ALL TRANSVERSE COMPONENTS OF $\mathbf{E}$ and $\mathbf{H}$ CAN BE DETERMINED FROM ONLY $E_z$ and $H_z$.

Since $E_z$ and $H_z$ each have its own wave equation, we can solve one assuming the other is zero and vice versa. The separate solutions can then be added up to give a general solution.

Then the solutions divide naturally into two families:

1. When $E_z = 0$ - Transverse Electric (TE)

2. When $B_z = 0$ - Transverse Magnetic (TM)

To solve the equations, we need boundary conditions, assuming perfect conductor:

1. $\mathbf{E}$ perpendicular to surface. So $E_z = 0$. (Used to solve $E_z$ equation. So $B_z = 0$ - TE.)

2. $\mathbf{B}$ parallel to the surface. So $\partial B_z / \partial n = 0$ - $n$ means normal to surface, e.g $x$ or $y$. (Used to solve $B_z$ equation. So $E_z = 0$ - TE.)

E.g. consider TE in rectangular waveguide. Wave equation for $B_z$ as same form as $E_x$ equation from earlier slide:

$$\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \frac{\partial^2 B_z}{\partial z^2} = \epsilon\mu \frac{\partial^2 B_z}{\partial t^2}$$

Try separable variables:

$$B_z = B_0(x, y) e^{ikz - \omega t}$$

Dropped subscript for $k_z$ as $z$ is only propagating direction possible. Substituting into wave equation gives

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \mu\epsilon\omega^2 - k^2 \right) B_z = 0$$

Applying boundary condition, find

$$B_z = H_0 \cos\left(\frac{m\pi x}{a}\right) \cos\left(\frac{n\pi y}{b}\right)$$

where $m, n$ are any integers. Substituting into wave equation gives

$$\mu\epsilon\omega^2 - k^2 = \pi^2 \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)$$

For sin($kz - \omega t$) to be a wave, $|k| > 0$. When $k = 0$, this equation gives the smallest $\omega$ - cutoff frequency.

Wave for each possible set of $m, n$ is called a mode.

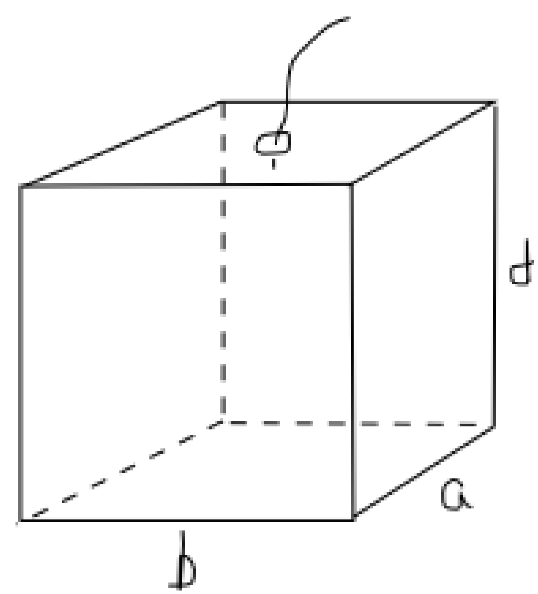
E.g. $a = 2b$, cutoff frequencies ratios (Jackson, 3rd ed.):

| $m$ \ $n$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| 0 | | 2 | 4 | 6 |
| 1 | 1 | 2.24 | 4.13 | |
| 2 | 2 | 2.84 | 4.48 | |
| 3 | 3 | 3.61 | 5 | |
| 4 | 4 | 4.48 | 5.66 | |
| 5 | 5 | 5.39 | | |
| 6 | 6 | | | |

In a waveguide, waves can be produced using a small antenna or coupler that sticks into the guide. Any mode with cutoff frequency lower than input frequency can be produced.

When an electron goes round the ring, it loses energy to radiation. When it passes through an rf cavity, it experiences an electric field in the forward direction. This gives it a push and replaces the energy that it has lost to radiation.

Consider a metallic, rectangular box. Make a tiny hole on one side of the box, and stick a tiny wire through it.

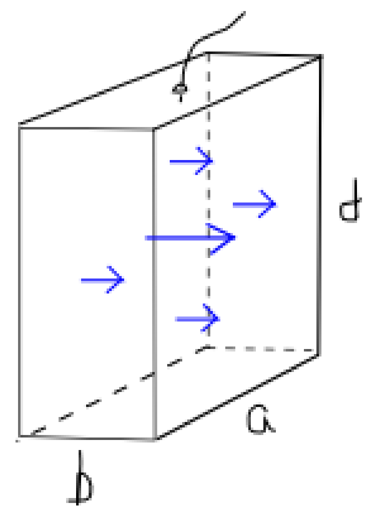


Apply an oscillating voltage to this wire. Electric field from current would then induces charges on inner walls of box. There is a natural frequency of oscillation for field and charges. Field can resonate and grow stronger and stronger - if the voltage on that wire through the hole has the same frequency.

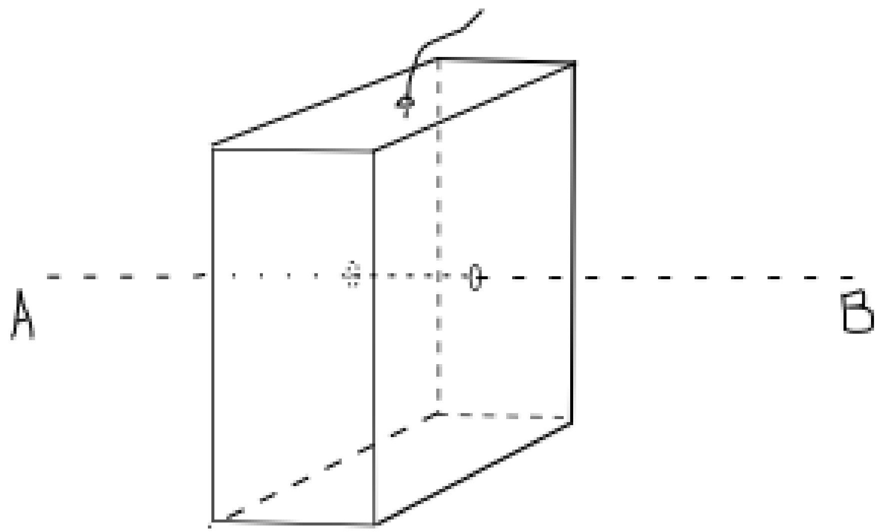
This resonant frequency is obtained by solving the wave equation. For a rectangular box, the wave equation can solved using a product of sine or cosine functions. The result is a standing wave with frequency:

$$f_{mnl} = \frac{c}{2\pi}\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2 + \left(\frac{l\pi}{d}\right)^2}$$

where $l, m, n$ are integers. Choose one side, say $b$, to be shorter than the other two. Choose $m = 1, n = 0, l = 1$.




This choice gives an oscillating electric field that is always parallel to the side b. It changes direction at the oscillation frequency, but must remain parallel to $b$. It is uniform along direction $b$.




Now make two small holes on the two opposite faces. Can pass an electron into one hole, and the field would accelerate the electron to the other hole! If AB is the desired path, just put the box in the path so that the path goes through the holes, and it will do the job of accelerating the particle.

Note that a cavity that uses the wave this way is called standing wave cavity. There are other ways, e.g. travelling wave cavity.

Gaussian units is based on cgs (centimetre-gram-second) units. It is commonly used in electromagnetism to make equations look simpler. But this also means that conversion to SI units is tricky because the equation must also be changed.

$c$ = 29,979,245,800 ≈ $3 \cdot 10^{10}$

| Quantity | Symbol | SI unit | Gaussian unit |
|---|---|---|---|
| electric charge | $q$ | 1 C | ↔ $(10^{-1}\ c)$ Fr |
| electric current | $I$ | 1 A | ↔ $(10^{-1}\ c)$ Fr/s |
| electric potential<br>voltage | $\varphi$<br>$V$ | 1 V | ↔ $(10^{8}\ c^{-1})$ statV |
| electric field | **E** | 1 V/m | ↔ $(10^{6}\ c^{-1})$ statV/cm |
| magnetic induction | **B** | 1 T | ↔ $(10^{4})$ Gs |
| magnetic field strength | **H** | 1 A/m | ↔ $(4\pi\ 10^{-3})$ Oe |
| magnetic dipole moment | **μ** | 1 A·m² | ↔ $(10^{3})$ erg/Gs |
| magnetic flux | $\Phi_m$ | 1 Wb | ↔ $(10^{8})$ Gs·cm² |
| resistance | $R$ | 1 Ω | ↔ $(10^{9}\ c^{-2})$ s/cm |
| resistivity | $\rho$ | 1 Ω·m | ↔ $(10^{11}\ c^{-2})$ s |
| capacitance | $C$ | 1 F | ↔ $(10^{-9}\ c^{2})$ cm |
| inductance | $L$ | 1 H | ↔ $(10^{9}\ c^{-2})$ s²/cm |

| Name | Gaussian units | SI units |
|---|---|---|
| Lorentz force | $\mathbf{F} = q\left(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}\right)$ | $\mathbf{F} = q\left(\mathbf{E} + \mathbf{v} \times \mathbf{B}\right)$ |
| Coulomb's law | $\mathbf{F} = \frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$ | $\mathbf{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}$ |
| Gauss's law (microscopic) | $\nabla \cdot \mathbf{E} = 4\pi\rho$ | $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ |
| Gauss's law for magnetism: | $\nabla \cdot \mathbf{B} = 0$ | $\nabla \cdot \mathbf{B} = 0$ |
| Maxwell–Faraday equation (Faraday's law of induction): | $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$ | $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ |
| Ampère–Maxwell equation (microscopic): | $\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t}$ | $\nabla \times \mathbf{B} = \mu_0\mathbf{J} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$ |

`http://en.wikipedia.org/wiki/Gaussian_units`

Google “Gaussian units” for details on conversions.

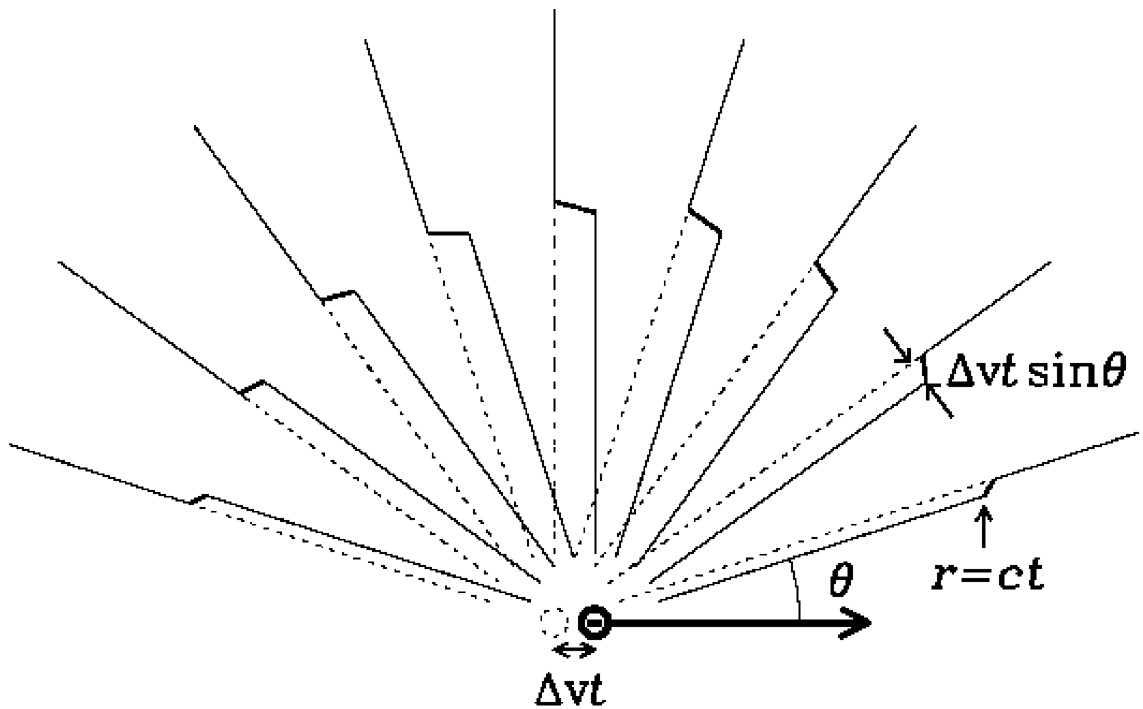
[In this topic, Gaussian units are used.]

When an electron passes through a magnet, it bends and radiates energy. This is called synchrotron radiation. The rate of radiation loss for an electron when it revolves around a circle is

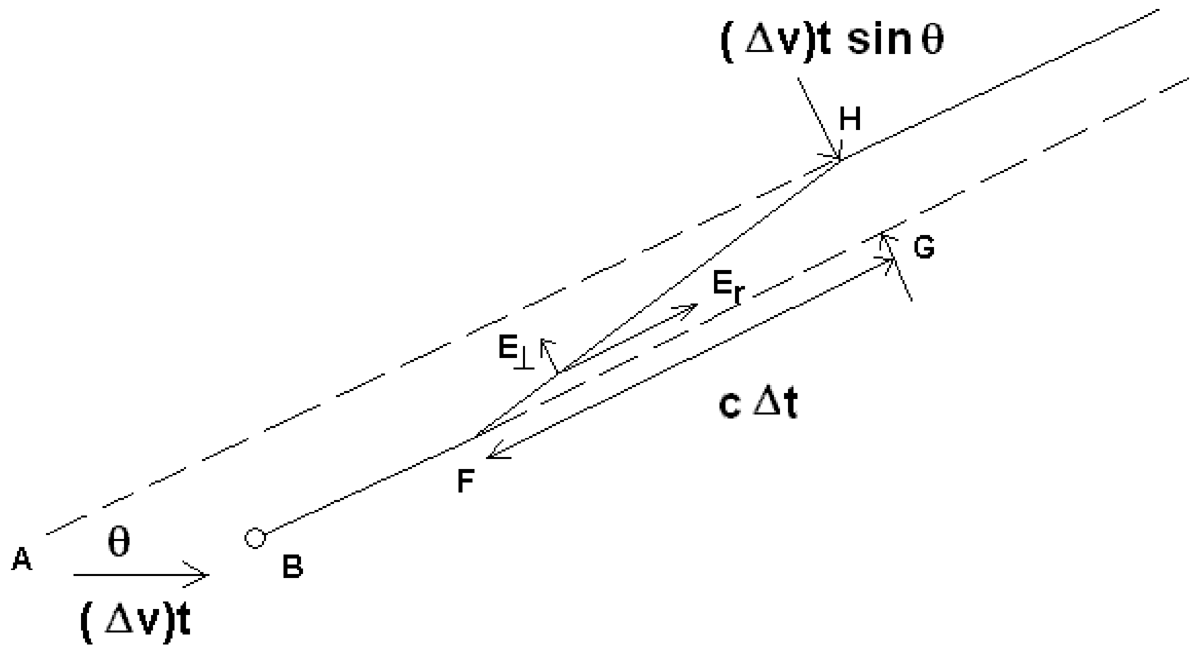
$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^4 a^2 \text{ [Gaussian units]}$$

(Gaussian units) where $q$ is electron charge, $a$ is acceleration and $\gamma$ is Lorentz factor.

This is Lienard's formula. To derive this and understand the basic physics, first derive the Larmor radiation formula.




An electron is accelerated for a short time $\delta t$ to a velocity $\delta v$. This "jerk" creates a kink in the electric field. This kink travels outward at the speed of light, and forms the radiation that is generated.




To estimate this transverse electric field component, consider the kink at a time $t$. The kink is just a slight distortion of the electric field. The magnitude of the electric field along the kink remain approximately the same as the Coulomb field of the electron at rest: $q/r^2$ [Gaussian units]. The radial component of the field is also approximately the same, i.e.

$$E_r = \frac{q}{r^2} \text{ [Gaussian units]}$$

With the help of the triangle FGH enclosing the kink, and using similar triangles, we can see that

$$\frac{E_\perp}{E_r} = \frac{(\Delta v)t \sin\theta}{c\Delta t} \text{ [Gaussian units]}$$

Note that the distance AB is approximated as $(\Delta v)t$ because the distance travelled during the initial $\Delta t$ has been neglected. Substituting $t = r/c$, the transverse component can now be obtained:

$$E_\perp = \frac{qa \sin\theta}{rc^2} \text{ [Gaussian units]}$$

where the acceleration $a = \Delta v/\Delta t$. Using the Poynting vector

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{E} \times \mathbf{B} \text{ [Gaussian units]}$$

and taking $E = B$ (cgs units for plane wave, true if the particle is oscillating sinusoidally), so that

$$S = \frac{c}{4\pi}E^2 \text{ [Gaussian units]}$$

the energy flux in direction $\theta$ can be derived:

$$S = \frac{1}{4\pi}\frac{q^2a^2}{c^3}\frac{\sin^2\theta}{r^2} \text{ [Gaussian units]}$$

Note that this result predicts that radiation is maximum in a direction perpendicular to the acceleration of the electron. The total power can now be obtained by integrating over all direction. This gives:

$$P = \frac{2}{3}\frac{q^2a^2}{c^3} \text{ [Gaussian units]}$$

This is Larmor's equation.

To derive the final expression, we show that when an electron goes round a circle at relativistic speed, the acceleration in Larmor's equation is increased by a factor of $\gamma^2$.

Suppose that in the lab frame, the electron is is moving in the $x$ direction at a particular instance in time, and accelerating in

the $y$ direction. In the rest frame, the electron velocity would be zero, but it would still be accelerating in the $y$ direction. This is where Larmor's equation is valid.

Then we do a Lorentz transformation to the lab frame. The simplest way to understand this is from the definition of acceleration: $a = d^2x/dt^2$. Because of time dilation, the $dt^2$ in the denominator gives the factor of $\gamma^2$.

Since the acceleration, $a$, gains a factor of the $\gamma^2$ when we transform to the lab frame, we can insert these into Larmor's equation and get

$$P = \frac{2}{3}\frac{q^2}{c^3}\gamma^4 a^2 \text{ [Gaussian units]}$$

This is Lienard's formula for the power from synchrotron radiation.

[We switch back to SI units here.]

The static electric potential $\phi$ is defined by

$$\mathbf{E} = -\nabla\phi$$

It is simplifies calculations and is interpreted as work done on a charge. The magnetic vector potential $\mathbf{A}$ is defined by

$$\mathbf{B} = \nabla \times \mathbf{A}$$

It also simplifies calculations but is not normally given a physical interpretation. It is often remembered as "the thing whose curl is magnetic field". It was historically interpreted as electromagnetic field momentum per unit charge. Lets see how.

First, note that when the above expression is substituted into Faraday's law

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$$

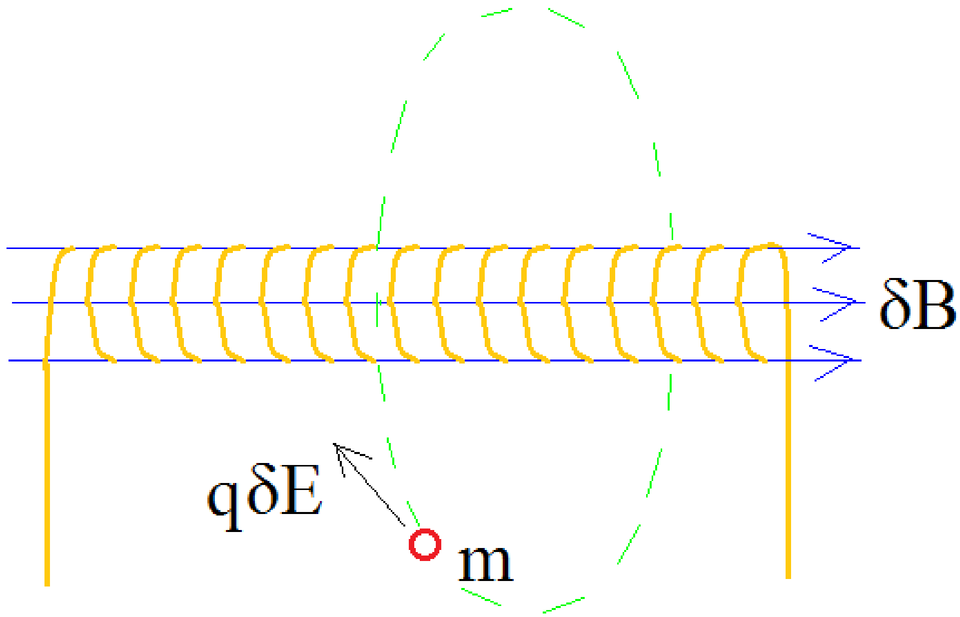
we get

$$\nabla \times \mathbf{E} = -\frac{\partial(\nabla \times \mathbf{A})}{\partial t} = \nabla \times \left(-\frac{\partial \mathbf{A}}{\partial t}\right)$$

This means that in time varying field, $\mathbf{E}$ has the additional term $-\partial\mathbf{A}/\partial t$. So static electric potential must be generalised to

$$\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$$

To show that $\mathbf{A}$ is field momentum per unit charge, consider a long solenoid with a charge $q$ outside.




If the magnetic field $\mathbf{B}$ changes over a short time, an electric field is generated around the solenoid according to Faraday's law:

$$\int_C \mathbf{E}.d\mathbf{l} = -\frac{d}{dt}\int_S \mathbf{B}.d\mathbf{S}$$

The vector potential definition can be written using Stoke's law as

$$\int_C \mathbf{A}.d\mathbf{l} = \int_S \mathbf{B}.d\mathbf{S}$$

Comparing the two equations for a short time $dt$,

$$\mathbf{E}dt = -d\mathbf{A}$$

$$-d(q\mathbf{A}) = q\mathbf{E}dt = \mathbf{F}dt = d(m\mathbf{v})$$

So an increase in particle momentum means a decrease in $q\mathbf{A}$. (Just like an increase in particle energy means a decrease in potential energy $q\phi$.)

So $q\mathbf{A}$ can be interpreted as field momentum.

We know that total kinetic energy and potential energy is conserved. In the same way, we expect $m\mathbf{v} + q\mathbf{A}$ to be conserved. So we can think of this as a generalised momentum

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}$$

Hamiltonian method is useful for calculations on particle motion in accelerators. The Hamiltonian is sum of kinetic and potential energy

$$H = T + V = \frac{1}{2}mv^2 + q\phi = \frac{(mv)^2}{2m} + q\phi$$

$T$ must be expressed in terms of momentum $p_i$. Since the system included both field and particle, $p_i$ should be the generalised momentum on the last slide

$$\mathbf{p} = m\mathbf{v} + q\mathbf{A}$$

This suggests that

$$H = \frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi$$

To check, must derive equation of motion from Hamilton's equations:

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i} \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}$$

In this case, $p_i$ is a component of $\mathbf{p}$ and $x_i$ is a component of $\mathbf{x}$. Lets start with $x$ components. First equation gives

$$\frac{dx}{dt} = \frac{\partial H}{\partial p_x} = \frac{p_x - qA_x}{m}$$
$$m\dot{\mathbf{x}} = \mathbf{p} - q\mathbf{A}$$

which is correct. Second equation gives:

$$\frac{dp_x}{dt} = -\frac{\partial}{\partial x}\left(\frac{(\mathbf{p} - q\mathbf{A})^2}{2m} + q\phi\right)$$
$$= -q\frac{(\mathbf{p} - q\mathbf{A})}{m}.\frac{\partial \mathbf{A}}{\partial x} - q\frac{\partial \phi}{\partial x}$$

Expanding

$$\frac{d(m\dot{x} + qA_x)}{dt} = -q\frac{(m\dot{\mathbf{x}})}{m}.\frac{\partial \mathbf{A}}{\partial x} - q\frac{\partial \phi}{\partial x}$$
$$\frac{dmv_x}{dt} = -q\frac{dA_x}{dt} - q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right) - q\frac{\partial \phi}{\partial x}$$

Using the full differential for $A_x$

$$\begin{aligned}
\frac{dmv_x}{dt} &= -q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_x}{\partial y} + \dot{z}\frac{\partial A_x}{\partial z} + \frac{\partial A_x}{\partial t}\right) \\
&\quad - q\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right) - q\frac{\partial \phi}{\partial x} \\
&= -q\left(-\frac{\partial A_x}{\partial y} + \frac{\partial A_y}{\partial x}\right) + q\left(-\frac{\partial A_z}{\partial x} + \frac{\partial A_x}{\partial z}\right) - qE_x - q\frac{\partial A_x}{\partial t}
\end{aligned}$$

Using definitions $\mathbf{B} = \nabla \times \mathbf{A}$ and $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$,

$$\begin{aligned}
\frac{dmv_x}{dt} &= qv_yB_z - qv_zB_y + qE_x \\
\frac{dm\mathbf{v}}{dt} &= q\mathbf{v} \times \mathbf{B} + q\mathbf{E}
\end{aligned}$$

We have recovered the Lorentz force equation. This shows that the Hamiltonian is correct.

# In Conclusion

We have

- reviewed the maths needed,
- derived Maxwell's equations,
- derived the wave equation,
- derived waveguide modes,
- derived synchrotron radiation,
- reviewed vector potential, and
- derived the Hamiltonian.